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ON ELASTIC CRACK-INCLUSION INTERACTION

I. KUNIN
Department of Mechanical Engineering, University of Houston, Houston, TX 77004, U.S.A.

and

B. GOMMERSTADT
Department of Mechanical Engineering, Northeastern University, Boston, MA 02115, U.S.A.

Abstract—The problem of a crack near an inclusion is treated using the projection integral equation method. An approximation for elastic fields and the energy of interaction is obtained. Relations between J- and M-integrals and those integrals with the interaction energy are established.

1. INTRODUCTION

The interaction of cracks with elastic inclusions is of importance in mechanics and micromechanics of fracture, the theory of dislocations, geomechanics, etc. Composite materials, such as ceramics and fiber reinforced composites, are heterogeneous solids consisting of homogeneous phases. The stress field around the crack tip at a bimaterial interface is drastically different than that for a homogeneous medium. A number of analyses attempted to predict the stress fields and stress intensity factors for cracks interacting with inclusions. Tamate[1] has treated the problem of a cracked sheet containing a circular elastic inclusion on the line of prolongation of a crack. The crack-inclusion interaction problem was considered also by Atcinson[2] and Erdogan *et al.*[3, 4].

Microscopic observations clearly indicate that in many cases cracks are quite different from an ideal cut. Typically, a zone of "damage" develops in the vicinity of the crack tip and accompanies the crack propagation[5, 6]. This zone can be identified and is visible as an entity adjacent to the crack. In most fracture mechanics approaches there is no explicit consideration of the damage zone and fracture process undergoing therein. At the same time, interaction between such damage and the main crack strongly effects the stress-strain field and related phenomena. Modeling of this interaction is presented in the concept of the "crack-layer" (Chudnovsky *et al.*[7]). The attention in this case should be concentrated on energy release consideration of the crack-inclusion system as a whole.

In this study, the problem of a crack with an inclusion is treated using the projection integral equations (PIE) method[8], which permits one to obtain approximate analytical or efficient numerical solutions for the inclusion problem. An approximation for the elastic fields and the energy of interaction between the crack and the small inclusion near the tip of the crack is obtained. Translational and expansional energy release rates for a crack-inclusion system are considered. Relations between these rates, J- and M-integrals, and the interaction energy are established.

2. THE GENERAL SCHEME OF THE PIE METHOD

Let us consider an unbounded elastic medium with the tensor of elastic moduli $C^{\lambda\alpha\mu\beta}(x)$, where x is a point of the medium and $\lambda, \ldots = 1, 2, 3$. Typically, the tensor function $C(x)$ is a constant or piecewise constant and may contain a δ-functional term corresponding to a crack. The equation for the static displacement $u(x)$ is

$$-\partial_\lambda C_0^{\lambda\alpha\mu\beta}(x)\partial_\mu u_\beta(x) = q^\alpha(x), \tag{2.1}$$

where $q(x)$ is the body force. The equation is to be understood in the sense of generalized functions. As a consequence, the continuity conditions at a surface where $C(x)$ has a jump are satisfied automatically. If $C(x)$ contains a δ-functional term, an additional regularization is necessary.

The eqn (2.1) can be written also in the compact form

$$Lu = q, \qquad L = -\partial C \partial. \tag{2.2}$$

The solution of this equation, which tends to zero at infinity, is expressed through the Green's tensor for the displacement

$$u_\alpha(x) = \int G_{\alpha\beta}(x, x')q^\beta(x')\,\mathrm{d}x', \tag{2.3}$$

or in compact form

$$u = Gq, \qquad GL = I, \tag{2.4}$$

where I is the identity operator.

Let the tensor function $C(x)$ admit a representation

$$C(x) = C_0(x) + C_1(x), \tag{2.5}$$

where the perturbation $C_1(x)$ is localized in a bounded domain V^+ with the characteristic function $V^+(x)$. In the case of an inclusion $C_1(x) = C_1V^+(x)$, where $C_1 =$ const. The case $C_1 \to \infty$ corresponds to a rigid inclusion, and $C_1 = -C_0$ corresponds to a void (in the latter case, C_0 is the constant tensor of elastic moduli of a homogeneous matrix).

The operator L is now represented in the form

$$L = L_0 + L_1, \quad L_0 = -\partial C_0 \partial, \quad L_1 = -\partial C_1 \partial, \tag{2.6}$$

and we assume that the Green's operator G_0 orresponding to L_0 ($G_0L_0 = I$) is known. Then the eqn (2.1) is equivalent to

$$u + G_0L_1u = u_0, \tag{2.7}$$

where $u_0 = G_0q$ is the external field, i.e. the displacement in the absence of the perturbation. Applying the symmetrized gradient operator def to (2.7) we obtain the integral equation for the strain $\epsilon = \operatorname{def} u$

$$\epsilon + K_0C_1\epsilon = \epsilon_0, \tag{2.8}$$

where $\epsilon_0 = \operatorname{def} u_0$, and

$$K_0 = -\operatorname{def} G_0 \operatorname{def} \tag{2.9}$$

is the Green's operator for strain in the medium with the elastic moduli $C_0(x)$. The kernel $K_0(x, x')$ of the operator K_0 is a generalized function defined by the corresponding regularization. Properties of the Green's function $K_0(x, x')$ are discussed in detail in [8].

Let $V^-(x)$ be the complement of $V^+(x)$, i.e. $V^+(x) = 1$, and let the corresponding multiplication operators be V^+ and V^-, i.e. $V^+ + V^- = I$. We define the operators

$$K_0^+ = V^+K_0V^+, \qquad K_0^- = V^-K_0V^+, \tag{2.10}$$

and let $\epsilon_0^+ = V^+\epsilon_0$, and so on, be the corresponding restrictions. Then eqn (2.8) is equivalent to the pair of equations

$$\epsilon^+ + K_0^+ C_1 \epsilon^+ = \epsilon_0^+, \tag{2.11}$$

$$\epsilon^- = \epsilon_0^- - K_0^- C_1 \epsilon^+. \tag{2.12}$$

The first equation determines the solution ϵ^+ inside the domain V^+ and the second determines its continuation on V^-.

Thus, for the perturbation localized in the domain V^+, the problem is reduced to the integral equation (2.11) inside V^+. The essence of the projection integral equation method is based on a correct definition of the singular operators K_0^+, K_0^-, in the sence of generalized functions[8].

The solution of the problem is equivalent to representing the Green's function G in the form

$$G = G_0 - G_0 \partial P_0 \partial G_0, \tag{2.13}$$

where the operator P_0 satisfies an integral equation in the domain V^+ and admits a representation

$$P_0 = -(C_1^{-1} + K_0^+)^{-1}. \tag{2.14}$$

It should be emphasized that, in contrast to the Green's operator G, the kernel of the operatort P_0 is localized in the bounded domain V^+ and its construction is facilitated by this both analytically and numerically.

The operator P_0 being given, the solution of eqn (2.11) has the form

$$\epsilon^+ = \epsilon_0^+ + K_0^+ P_0 \epsilon_0^+. \tag{2.15}$$

Let us now consider the total strain energy Φ

$$\begin{aligned} \Phi &= \frac{1}{2}\int q(x)U(x)\,\mathrm{d}x \\ &= \frac{1}{2}\iint q(x)G(x, x')q(x')\,\mathrm{d}x\,\mathrm{d}x'. \end{aligned} \tag{2.16}$$

Using (2.13) leads to the decomposition

$$\Phi = \Phi_0 + \Phi_{\mathrm{int}}. \tag{2.17}$$

The first term

$$\Phi_0 = \frac{1}{2}\iint q(x)G_0(x, x')q(x')\,\mathrm{d}x\,\mathrm{d}x' \tag{2.18}$$

is the strain energy of the external field itself, i.e. without the perturbation. The second term Φ_{int} is the interaction energy of the perturbation L_1 and the external strain ϵ_0 (or stress σ_0)

$$\begin{aligned} \Phi^{\mathrm{int}} &= \frac{1}{2}\iint \epsilon_0(x)P_0(x, x')\epsilon_0(x')\,\mathrm{d}x\,\mathrm{d}x' \\ &= \frac{1}{2}\iint \sigma_0(x)R_0(x, x')\sigma_0(x')\,\mathrm{d}x\,\mathrm{d}x', \end{aligned} \tag{2.19}$$

where

$$R_0 = C_0^{-1}P_0C_0^{-1}. \tag{2.20}$$

The operators P_0 and R_0 which define Φ_{int} are the interaction energy operators for strain and stress respectively.

If the domain V^+ is small, the perturbation L_1 can be considered as a force dipole

$$q^\nu(x) = -Q^{\nu\mu}\partial_\mu\delta(x - x_0), \tag{2.21}$$

where $x_0 \in V^+$. The induced dipole moment $Q^{\nu\mu}$ linearly depends on the external field

$$Q^{\nu\mu} = P^{\nu\mu\lambda\tau}\epsilon^0_{\lambda\tau}(x_0), \tag{2.22}$$

where the constant tensor $P^{\nu\mu\lambda\tau}$ has the symmetry of the tensor of elastic constants and is a functional of the operator P_0[8].

The interaction energy Φ_{int} depends on the position x_0 of the defect. The corresponding configurational force is

$$f_\lambda = -\frac{\partial}{\partial x_0^\lambda}\Phi_{int} = Q^{\nu\mu}\partial_\lambda\epsilon^0_{\nu\mu}(x_0), \tag{2.23}$$

where the second equality stands for the point defect. The force f_λ defines the energy release rate due to the translational motion of the defect.

3. THE CRACK IN THE ELASTIC MEDIUM

Let us consider a crack Ω in a homogeneous elastic medium with the tensor of elastic constants C_0. Let $n(x)$ be the normal to the crack surface Ω and $\sigma_0^+(x)$ the restriction of the external stress $\sigma_0(x)$ to the crack Ω. Then the displacement discontinuity $b(x)$ at Ω satisfies the integral equation[8]

$$\int_\Omega T(x, x')b(x')\,dx' = n(x)\sigma_0^+(x), \qquad x \in \Omega, \tag{3.1}$$

where the generalized tensor function $T(x, x')$ is closely related to the Green's operator for stress

$$S_0 = C_0 - C_0K_0C_0. \tag{3.2}$$

It is shown[8] that

$$T(x, x') = n(x)S_0(x - x')n(x'), \tag{3.3}$$

and the left hand side of (3.1) admits the following regularization:

$$(Tb)(x) = \int_\Omega T(x, x')[b(x') - b(x)]\,dx' - n(x)\Gamma(x)b(x), \tag{3.4}$$

where $\Gamma(x)$ is a tensor function which depends on Ω only.

The inverse operator T^{-1} is related to the operators R_0 and P_0 which define the interaction energy

$$T^{-1} = nR_0n. \tag{3.5}$$

Notice that the operator T can be expressed also through the so-called second Green's function[7].

As is well known, the asymptotic stress σ in the neighborhood of the crack tip is proportional to $1/\sqrt{r}$, where r is the distance from the crack tip. The corresponding coefficients are stress intensity factors closely related to energy release rates. All such quantities can be expressed through first moments of the multipole expansion of the operator R_0. These moments are calculated explicitly in the particular case of an elliptic crack[8].

4. CRACK AND INCLUSION

Now we apply the general scheme outlined in Section 2 to the elastic medium which contains both the crack and inhomogeneity (inclusion).

It is convenient to represent the operator L in the form

$$L = L_* + L_1, \qquad L_1 = -\partial C_1 \partial, \tag{4.1}$$

where L_* describes the medium with the crack, and $C_1(x)$ is a perturbation in elastic moduli due to the inclusion. Let G_* be the Green's operator corresponding to L_*. The Green's function $G_*(x, x')$ is known for certain two-dimensional problems.

The integral equation (2.11) becomes now

$$\epsilon^+ + K_*^+ C_1 \epsilon^+ = \epsilon_*^+, \tag{4.2}$$

where

$$K_*^+ = V^+ K_* V^+, \qquad K_* = -\text{def}\, G_*\, \text{def}, \tag{4.3}$$

V^+ is the inclusion domain, and ϵ_* is the strain in the absence of the inclusion.

The Green's operator G for the medium with both the crack and inclusion has the form

$$G = G_* - G_* \partial P_* \partial G_*, \tag{4.4}$$

where the operator P_* admits the representation (cf. (2.14))

$$P_* = -(C_1^{-1} + K_*^+)^{-1}. \tag{4.5}$$

The solution of (4.2) is now given by

$$\epsilon^+ = \epsilon_*^+ + K_*^+ P_* \epsilon_*^+. \tag{4.6}$$

The interaction energy Φ_{int} between the inclusion and the field ϵ_* is

$$\Phi_{int}^* = \frac{1}{2} \iint \epsilon_*^+(x) P_*(x, x') \epsilon_*^+(x')\, dx\, dx', \tag{4.7}$$

where $P_*(x, x')$ is the kernel of the operator P_*.

The total strain energy Φ is equal to the sum

$$\Phi = \Phi^* + \Phi_{int}^*, \tag{4.8}$$

where Φ^* is the strain energy of the field ϵ_* itself, i.e. in the medium with the crack only.

The total strain energy can also be written in the form

$$\Phi = \Phi^0 + \Phi_{\rm int}, \tag{4.9}$$

where Φ^0 is the strain energy of the external field ϵ_0 without the crack and inclusion, and $\Phi_{\rm int}$ is the total interaction energy of ϵ_0 with the crack-inclusion system. We have

$$\Phi_{\rm int} = \frac{1}{2}\iint \epsilon_0(x)P(x, x')\epsilon_0(x')\,dx\,dx'. \tag{4.10}$$

where P is the corresponding interaction energy operator for which the following expression is easily obtained:

$$P = P_0 + (I + P_0K_0^+)P(I + K_0^+P_0). \tag{4.11}$$

Thus, the total interaction energy $\Phi_{\rm int}$ is expressed as a sum of the two terms. The first term is the usual interaction energy of the external field ϵ_0 with the crack that is analyzed in fracture mechanics. The second term reflects an additional contribution due to interactions of the inclusion with the external field ϵ_0 and the crack. As was indicated above, the computation of this term is equivalent to solving the integral equation (4.2) in the domain of the inclusion. The problem is essentially simplified when the domain can be considered as small.

Assume, first, that a single inclusion is localized in a small domain in the neighborhood of point x_0. Then the operator C_1 for the elastic moduli of the inclusion can be approximated by a δ-functional model

$$C_1(x) = vC_1\delta(x - x_0), \tag{4.12}$$

where v is the volume of the domain, C_1 perturbation of the tensor of elastic constants. Note that a quasi-continuum with a characteristic parameter of the order of the inclusion size should be incorporated to make the δ-functional model mathematically correct[8]. The model (4.12) reduces (4.11) to

$$P = P_0 + (I + P_0g_0)P_*(I + g_0P_0), \tag{4.13}$$

where

$$P_* = -v(C_1^{-1} + vg_*)^{-1} = \text{const}, \tag{4.14}$$

and g_0, g_* are known constant tensors. Thus, the operator (4.13) is defined explicitly for this model (as distinct from (4.11)) if the operators G_* and P_* are known. The strain field ϵ and the interaction energy $\Phi_{\rm int}$ may be obtained using (2.15) and (2.19) where ϵ_0 and P_0 should be substituted by ϵ_* and P_*, respectively.

As an application, we consider the plane problem of the crack-inclusion interaction in an infinite isotropic medium with shear modulus μ_0 and Poisson's ratio ν_0. Let $2l$ be the crack length, a the characteristic size of the inclusion, and r the distance from the inclusion center to the crack ($r \ll l$). We introduce the dimensionless parameters

$$\bar{\mu} = \frac{\mu_1}{\mu_0}, \quad \eta = \frac{r}{l}, \quad \xi = \frac{a}{r} \tag{4.15}$$

and consider first the case of small inclusion, i.e. $\xi \to 0$. Then the δ-functional model (4.12) is appropriate and the total interaction energy $\Phi_{\rm int}$ takes the form

$$\Phi_{\rm int} = \Phi_{\rm int}^0[1 + \eta\xi^2\psi_0], \tag{4.16}$$

where

$$\psi_0 = \frac{\bar{\mu}(b_1 + b_2\bar{\mu})}{(1 + b_3\bar{\mu})(1 + b_4\bar{\mu})}, \tag{4.17}$$

and $b_i = b_i(\nu_0, \nu_1)$ are constants of the order of unity which are calculated explicitly for a given model.

Returning to the case of an arbitrary inclusion when ξ is not assumed to be small, we obtain an expression for Φ_{int} which is a generalization of (4.16)

$$\Phi_{int} = \Phi^0_{int}[1 + \eta\xi^2\psi_0(\xi)], \tag{4.18}$$

where $\psi(\xi)$ depends on the shape of the inclusion and $\psi(0) = \psi_0$. There are certain cases when $\psi(\xi)$ can be computed explicitly using results obtained for a crack interacting with another single crack[9, 10], with several cracks[7], and with an elastic cylindrical inclusion[1–4].

The interaction energy Φ^0_{int} of the constant external stress σ_0 and the single crack is a well known quantity. In particular if σ_0 is normal to the crack

$$\Phi^0_{int} = \pi l^2 \frac{1 - \nu_0}{2\mu_0}\sigma_0^2. \tag{4.19}$$

For arbitrary σ_0 = const, Φ^0_{int} is proportional to l^2 and σ_0^2.

5. ENERGY RELEASE RATES

For the model under consideration, $\Phi_{int} = \Phi_{int}(l, r, a)$. Consequently, three cases of the crack-inclusion motion can be distinguished:

(1) the crack length $2l$ is varying while r, a = const (translation of the system as a whole);
(2) the distance r is varying while l, a = const (relative translation of the inclusion);
(3) the inclusion size a is varying while l, r = const (swelling of the inclusion).

Let us consider the energy release rates corresponding to these three motions.

As is well known, the energy release rate for a single crack is described by the J-integral

$$J^0 = \frac{1}{2}\frac{\partial\Phi_{int}}{\partial l}. \tag{5.1}$$

Analogously we define for the first case

$$J = \frac{1}{2}\frac{\partial\Phi_{int}}{\partial l}, \tag{5.2}$$

for the second case

$$J^* = \frac{\partial\Phi_{int}}{\partial r} \tag{5.3}$$

and for the third case

$$M^* = a\frac{\partial\Phi_{int}}{\partial a}. \tag{5.4}$$

It can be proved that J, J^*, and M^* are path-independent integrals corresponding to conservation laws with respect to translations and expansion.

Using (4.18) we find[11]

$$J = J^0[1 + \tfrac{1}{2}\eta\xi^2\psi(\xi)], \tag{5.5}$$

$$J^* = J^0[\xi^2\psi(\xi) + \xi^3\psi'(\xi)], \tag{5.6}$$

$$M^* = J^0 r[2\xi^2\psi(\xi) + \xi^3\psi'(\xi)]. \tag{5.7}$$

Excluding $\psi(\xi)$ from (4.18) and (5.5–5.7) permits one to establish the important relations between the interaction energy and the energy release rates

$$\Phi^*_{\text{int}} = M^* + rJ^*, \tag{5.8}$$

$$J = l^{-1}(\Phi^0_{\text{int}} + \tfrac{1}{2}\Phi^*_{\text{int}}) = J^0 + \frac{r}{2l}J^* + \frac{1}{2l}M^*. \tag{5.9}$$

It is readily seen that J is a superposition of energy release rates due to the absolute and relative translations, and expansion. The contribution of the last two terms to J depends on the shape, location, and rigidity of the inclusion. As a rule, for soft (rigid) inclusions the contribution will be positive (negative). Under certain conditions a screening effect can be observed[7].

Notice that one may try to interpret (5.8) as

$$M = M^* + rJ^*, \tag{5.10}$$

where M is a new dilatation energy release rate corresponding to a new coordinate system origin. However, M would not satisfy the relation analogous to (5.4), i.e. M is not a true energy release rate, but rather a combination of expansional and translational modes.

6. GENERAL RELATIONS BETWEEN J, M, AND Φ_{int}

The relations between the energy release rates and the interaction energy obtained above were due to the special functional dependence of Φ_{int} on parameters l, r, a which is reflected in (4.18). This is a consequence of the assumption that the inclusion is located in a neighborhood of the crack tip where the stress is proportional to $r^{-1/2}$. We consider now certain general relations which do not depend on assumptions of such a type[12].

Let us start from the definition of the J- and M-integrals[13, 14]

$$J_i = \int_S P_{ij}n_j \, \mathrm{d}S, \qquad M_0 = \int_S x_i P_{ij} n_j \, \mathrm{d}S, \tag{6.1}$$

where P_{ij} is an energy-momentum tensor, n_i the normal to a contour S surrounding the inclusion, x_i components of a position vector $\mathbf{r}$ with respect to an origin O, and $i, j = 1, 2$. We can identify S with the boundary of the inclusion domain V without loss of generality. Note that M_0 depends on location of the origin.

It is convenient to rewrite (6.1) by means of volume integrals. Let

$$q_i = \partial_j P_{ij}, \qquad p = P_{ii}. \tag{6.2}$$

Then we have

$$\mathbf{J} = \int_V \mathbf{q} \, \mathrm{d}V, \tag{6.3}$$

$$M_0 = \int_V p \, \mathrm{d}V + \int_V \mathbf{r}\cdot\mathbf{q} \, \mathrm{d}V. \tag{6.4}$$

These equations show that if $\mathbf{J} = 0$, the integral M_0 is independent of the choice of the origin O. If $\mathbf{J} \neq 0$ and x_2-axis is orthogonal to $\mathbf{J}$, the following statements can be proved: (a) M_0 is invariant with respect to translation along the x_2-axis; and (b) there exists a unique position of the origin at the x_1-axis for which the second integral (6.4) is equal to zero.

A coordinate system satisfying (b) will be called a scale coordinate system. All these systems are defined up to translation in the x_2-direction.

The J- and M-integrals are related to energy release rates[15]. Let Φ_{int} be the energy of interaction of the inclusion with σ_0 and a, a_i, be parameters of self-similar expansion and translation, respectively. Then

$$J_i = \frac{\partial}{\partial a_i}\Phi_{int}, \qquad M_0 = a\frac{\partial}{\partial a}\Phi_{int}. \tag{6.5}$$

It follows from the Noether theorem [14, 16] and the statements (a) and (b) that the second expression in (6.5) is valid only for $\mathbf{J} = 0$ or a scale coordinate system. To our knowledge, this has not been stated before.

In what follows we assume that σ_0 = constant. Then it can be shown that

$$\mathbf{J} = 0, \qquad M \equiv M_0 = 2\Phi_{int}. \tag{6.6}$$

The first equality is evident. To prove the second one it is convenient to take M in the form (6.1) with an infinitely large contour S. Then it is possible to substitute an equivalent point defect for the inclusion[8], and, as a result, to prove Φ_{int} to be a homogeneous function of a of the second degree.

Let us formally divide an inclusion V into parts V_1 and V_2, $V = V_1 + V_2$. Then

$$\mathbf{J}_1 + \mathbf{J}_2 = 0, \tag{6.7}$$

where $\mathbf{J}_1$ and $\mathbf{J}_2$ are J-integrals for V_1 and V_2, respectively.

Now we introduce scale coordinate systems for each inclusion and let $\mathbf{r}_{12}$ be the vector from the origins O_1 to O_2, and M_1, M_2 be the corresponding M-integrals for V_1 and V_2. Then, from the previous formulas, we can easily deduce the following relations:

$$M = M_1 + M_2 + \mathbf{r}_{12}\cdot\mathbf{J}_1 = 2\Phi_{int}. \tag{6.8}$$

In the particular case when V is a crack of the length $2l$, we have $M_1 = M_2 = 0$ and (6.8) reduces to the well-known relation[13]

$$M^0 = 2lJ^0.$$

Employing this relation, we rewrite the formula (5.9) in the form

$$M^0 + M^* + rJ^* = 2\Phi_{int}, \tag{6.9}$$

which is a particular case of the general formula (6.8).

Notice that M_1, M_2, and $\mathbf{J}_1$ in (6.8) are defined as the corresponding path-independent integrals and no assumptions about the distance between inclusions and their shapes are necessary.

5. CONCLUSION

The results obtained here can be extended to more general models. Localized defects can be not only inhomogeneities, but can, in addition, also include a source of internal (residual) stress. To extend the presented approach to this general case an appropriate renormalization of the effective characteristics of the inclusion is necessary.

An extension of the proposed approach to a system of defects is also possible. In this case a solution is not a superposition of perturbations caused by individual defects because of interactions between defects. The tensor describing corresponding interactions can be calculated explicitly for δ-functional models.

In the general case, the kernel of the operator P might be found by numerical methods. Use of the latter is facilitated by the fact that, unlike Green's functions, the operator P is localized in defect domains. Numerical methods are applied directly to integral equations similar to eqn (2.11).

Note that the formula (6.8) can be easily generalized for a multi-inclusion problem.

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